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Lattice models of branched polymers: effects of geometrical constraints

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Abstract. We consider uniform brushes on subsets of the square and simple cubic lattices. We show that, in a slab geometry in three dimensions, the connective constant of a brush is identical to that of a walk in that slab. In two dimensions, in a slit geometry, the connective constant of a brush is strictly less than that of a walk. We also consider the intermediate case of a rectangular prism and show that the connective constant of a brush is strictly less than that of a walk; similar results are presented for combs, stars and polygons.

1. Introduction

Two areas of recent interest have been the properties of uniform branched polymers (Miyake and Freed 1983, Vlahos and Kosmas 1984, Duplantier 1986, Lipson *et al* 1987) and the effect of geometrical constraints on the properties of self-avoiding walks (Wall and Klein 1979, Klein 1980, Hammersley and Whittington 1985, Chayes and Chayes 1986), polygons (Klein 1980, Hammersley and Whittington 1985, Soteros and Whittington 1988) and uniform branched polymers (Duplantier and Saleur 1986, Colby *et al* 1987, Chee and Whittington 1987, Soteros and Whittington 1988). In earlier papers we have focused on the connective constant of a uniform star in a slab geometry (in three dimensions) and in a slit geometry (in two dimensions). We showed that the connective constant of a uniform star in a slab is identical to that of a self-avoiding walk in a slab (Chee and Whitttington 1987) but that the connective constant of a star (and, incidently, a polygon) in a slit is strictly less than that of a self-avoiding walk in a slit (Soteros and Whittington 1988).

In this paper we extend these results in two ways. A tree is a *brush* if and only if there is a self-avoiding walk which is a subgraph of the tree and which connects all the branch points (the vertices of degree greater than 2) of the tree. A *uniform brush* has the same number of edges in each branch. We distinguish different uniform brushes by $\{v_i\}$, the ordered set of degrees of the branch points and by n, the number of edges in each branch. We shall be concerned with the number of weak embeddings of these graphs in subsets of the square and simple cubic lattices. For instance, for the complete square lattice, the number of brushes with two branch points, $v_1 = v_2 = 3$, $n = 1$, is 18, and the number with two branch points, $v_1 = 3$, $v_2 = 4$, $n = 1$ is 12. A star is the special case of a brush with one branch point and a comb is the special case with all v_i equal to 3.

We prove that the connective constant of a uniform brush in $\mathbb{Z}^d$ is equal to that of a self-avoiding walk.

An *L-slab in* $\mathbb{Z}^3$ is that part of the lattice lying in and between the two planes $z = 0$ and $z = L$, and we prove that the connective constant of a uniform brush in this lattice subset is equal to that of a self-avoiding walk in this subset. An *L-slit in* $\mathbb{Z}^2$ is that part of the lattice lying in and between the two lines $y = 0$ and $y = L$. In this case we prove that the connective constant of a uniform brush in a slit is strictly less than that of a self-avoiding walk in a slit.

We also extend these results to an infinite right prism, i.e. that part of $\mathbb{Z}^3$ lying in and between the four planes $y = 0$, $y = L_1$, $z = 0$ and $z = L_2$. We prove that the connective constant of a brush in this lattice subset is strictly less than that of a self-avoiding walk in the same subset, and we prove a corresponding result for polygons.

2. Brushes in a slab

In this section we prove that uniform brushes in a slab in $\mathbb{Z}^3$ have the same connective constant as self-avoiding walks in the same slab. We first prove that suitably unfolded stars in a slab have the same connective constant as walks. Concatenating these stars to form brushes gives a lower bound on the connective constant of a brush. An upper bound is obtained by connecting walks to form a set of graphs which contains the brushes.

The proofs rely heavily on the unfolding argument of Hammersley and Welsh (1962) and on results for the connective constant of walks in wedges (Hammersley and Whittington 1985, Whittington 1988). In particular, the proofs will rely on concatenating unfolded walks in the wedges, $W_1, \ldots, W_5$, of figure 1 to create unfolded stars. Hence we need to define appropriate unfolding operations in these wedges and determine the connective constant of unfolded walks in these wedges.

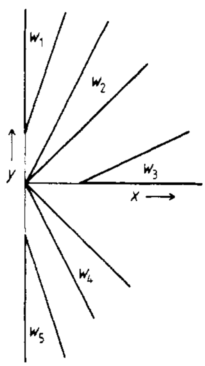

Figure 1. Projections in the xy plane of the wedges $W_1, \ldots, W_5$.

Write (x_i, y_i, z_i) for the coordinates of the ith vertex of a walk having n steps. We say that a walk is *unfolded in the* γ *direction* if, for all i,

$$\gamma y_0 + x_0 \leq \gamma y_i + x_i \leq \gamma y_n + x_n \tag{2.1}$$

where $0 \leq \gamma < \infty$. When $\gamma = 0$ this is equivalent to the definition of Hammersley and Welsh (1962).

Define an (α, β, L)-wedge, $\beta > \alpha$, as the subset of $\mathbb{Z}^3$ bounded by the four planes $z = 0$, $z = L$, $y = \alpha x$, $y = \beta x$, with $x \geq 0$. Choose any point $\boldsymbol{q}_0 = (x_0, y_0, z_0)$ which lies in the (α, β, L)-wedge such that there is at least one infinitely long self-avoiding walk in the wedge having this point as origin. Then the following lemma holds.

Lemma 2.1. Let $c_n^+(\alpha, \beta, \gamma, L, \boldsymbol{q}_0)$, $\beta > \alpha$, be the number of n-step self-avoiding walks starting at $\boldsymbol{q}_0 = (x_0, y_0, z_0)$ confined to an (α, β, L)-wedge, and unfolded in the γ direction. Then if (i) $\alpha = \gamma = 0$, $\beta > 0$ or (ii) $0 < \alpha \leq 1$, $\beta \geq 1$, $\gamma = 1$,

$$\lim_{n \to \infty} n^{-1} \log c_n^+(\alpha, \beta, \gamma, L, \boldsymbol{q}_0) = \kappa(L) \tag{2.2}$$

where $\kappa(L) \equiv \lim_{n \to \infty} n^{-1} \log c_n(L, z_0)$ and $c_n(L, z_0)$ is the number of n-step self-avoiding walks confined to an L-slab, starting at $(0, 0, z_0)$, for any z_0 such that $0 \leq z_0 \leq L$.

Proof. For $\alpha = \gamma = 0$ the proof is a straightforward extension of the ideas in section 4 of Hammersley and Whittington (1985) and Whittington (1988). For $\gamma = 1$ a walk can be unfolded by successive reflections of parts of the walk in planes $y + x =$ constant. Each such reflection gives a walk in the lattice and the reflections are continued until equation (2.1) is satisfied. Such walks can be concatenated and have similar properties to those with $\gamma = 0$. The remainder of the proof follows that of Whittington (1988).

We consider a uniform star with f branches each containing n edges, having its branch point (vertex of degree $f \leq 5$) at the point $(0, 0, z_0)$, $0 \leq z_0 \leq L$, and write (x_{ij}, y_{ij}, z_{ij}) for the coordinates of the ith vertex in the jth branch. The star is *unfolded in an L-slab* if

(i) $0 \leq z_{ij} \leq L$

(ii) $x_{ij} \geq 0$

(iii) there exists k such that $x_{ij} < x_{nk}$ for $i \neq n, j \neq k$

(iv) $0 < z_{nk} < L$.

We write $s_n^+(f, L, z_0)$ for the number of such unfolded stars.

Lemma 2.2.

$$\lim_{n \to \infty} (fn)^{-1} \log s_n^+(f, L, z_0) = \kappa(L). \tag{2.3}$$

Proof. We prove equation (2.3) for the case $0 < z_0 < L$. For $f < 5$ and $z_0 = 0$ or $z_0 = L$ the proof is similar.

By concatenating f walks of n steps, each lying in an L-slab, such that they have common origin $(0, 0, z_0)$, we obtain the upper bound

$$s_n^+(f, L, z_0) \leq c_n(L, z_0)^f. \tag{2.4}$$

We define five disjoint, translated, (α, β, L)-wedges as follows:

W_1 :	$x \geq 0$	$y \geq 3x + 4$	$0 \leq z \leq L$
W_2 :	$x \geq 0$	$x \leq y \leq 2x$	$0 \leq z \leq L$
W_3 :	$x \geq 4$	$0 \leq y \leq \frac{1}{2}(x - 4)$	$0 \leq z \leq L$
W_4 :	$x \geq 0$	$-2x \leq y \leq -x$	$0 \leq z \leq L$
W_5 :	$x \geq 0$	$y \leq -3x - 4$	$0 \leq z \leq L.$

In order to create a uniform unfolded star we shall concatenate five walks, one in each of the five wedges, and then unfold (in the x direction) the walk (in W_k) which extends the farthest in the positive x direction. This walk may then extend outside W_k but the resulting star will be self-avoiding. In fact it will be convenient to concatenate five walks which are unfolded in their respective wedges so that, after the final unfolding of the walk in W_k, we can force $0 < z_{nk} < L$.

W_1 is a translated $(3, \infty, L)$-wedge and so a translated and reflected $(0, \frac{1}{3}, L)$-wedge. Hence the number of n-step walks in W_1, unfolded along the positive y direction and starting at $(0, 4, z_0)$, $c_n^+(W_1)$ is equal to $c_n^+(0, \frac{1}{3}, 0, L, \boldsymbol{q}_0)$ where $\boldsymbol{q}_0 = (0, 0, z_0)$. Similarly, the number of n-step walks in $W_2, \ldots, W_5$, unfolded along $y = x$, $y = 0$, $y = -x$, and the negative y direction, respectively, and starting at $(1, 2, z_0 + 1)$, $(4, 0, z_0)$, $(1, -2, z_0 - 1)$ and $(0, -4, z_0)$, respectively, are given by $c_n^+(W_2) = c_n^+(1, 2, 1, L, \boldsymbol{q}_1)$, $c_n^+(W_3) = c_n^+(0, 1/2, 0, L, \boldsymbol{q}_0)$, $c_n^+(W_4) = c_n^+(1, 2, 1, L, \boldsymbol{q}_2)$, $c_n^+(W_5) = c_n^+(0, \frac{1}{3}, 0, L, \boldsymbol{q}_0)$, respectively, with $0 < z_0 < L$, $\boldsymbol{q}_1 = (1, 2, z_0 + 1)$ and $\boldsymbol{q}_2 = (1, 2, z_0 - 1)$.

If we define u_1, u_2, u_3 as unit steps along the positive x, y and z directions, and $\bar{u}_1$, $\bar{u}_2$, $\bar{u}_3$ as corresponding steps along the negative directions, we can join the vertex $(0, 0, z_0)$ to a vertex in each of these wedges by the following disjoint links: $l_1 = u_2^4$, $l_2 = u_3 u_2^2 u_1$, $l_3 = u_1^4$, $l_4 = \bar{u}_3 \bar{u}_2^2 u_1$, $l_5 = \bar{u}_2^4$.

We now concatenate the links $l_1, \ldots, l_f$ and f suitably unfolded walks, each of $n-7$ edges, in the wedges $W_1, \ldots, W_f$. The resulting graphs are uniform stars with $n-3$ edges in each of the f branches. Choose the wedge (k, say) which contains the vertex having maximum x coordinate amongst all the vertices in the uniform star. In case of ambiguity choose k to be first in the sequence $3, 2, 4, 1, 5$. Unfold the walk in W_k in the positive x direction (by reflections through planes $x =$ constant) and add to this unfolded walk the three steps u_1^3 if the z coordinate of the last vertex is neither 0 nor L, $u_1 \bar{u}_3 u_1$ if the z coordinate is L and $u_1 u_3 u_1$ if the z coordinate is 0. The walk in W_3 can always be extended by the three steps u_1^3, the walk in W_1 (W_5) can always be extended by the three steps u_2^3 ($\bar{u}_2^3$) and the walk in W_2 (W_4) can be always extended by adding one of the four subwalks $u_1 u_2 u_3$, $u_1 u_2 \bar{u}_3$, $u_2 u_1 u_3$, $u_2 u_1 \bar{u}_3$ ($u_1 \bar{u}_2 u_3$, $u_1 \bar{u}_2 \bar{u}_3$, $\bar{u}_2 u_1 u_3$, $\bar{u}_2 u_1 \bar{u}_3$). Extend the walks in each of the wedges W_j, $j \neq k$, in this way. The resulting star with n edges in each branch has a unique rightmost vertex and is therefore unfolded. Hence

$$s_n^+(f, L, z_0) \geq \exp(\mathrm{O}(\sqrt{n})) \prod_{i=1}^{f} c_{n-7}^+(W_i) \tag{2.5}$$

where the $\exp(\mathrm{O}(\sqrt{n}))$ term comes from the final unfolding in the x direction. Equation (2.5) together with lemma (2.1) and equation (2.4) gives equation (2.3).

An immediate corollary of this lemma is that uniform stars in a semi-infinite L-slab, $\{(x, y, z) : x \geq 0, 0 \leq z \leq L\}$, with their branch point in the plane $x = 0$ have connective constant $\kappa(L)$.

We now prove the main result of this section. Let $b_n(t, v_1, v_2, \ldots, v_t, L)$ be the number of uniform brushes in an L-slab having t branch points, of vertex degree $v_1, v_2, \ldots, v_t$, and n edges in each branch. Two brushes are counted as distinct if they cannot be superimposed by translation.

Theorem 2.1. For any t, $v_1, \ldots, v_t$, such that $v_i \leq 6$ for all i,

$$\lim_{n \to \infty} \left[n \left(1 - t + \sum_{i=1}^{t} v_i \right) \right]^{-1} \log b_n(t, v_1, v_2, \ldots, v_t, L) = \kappa(L). \tag{2.6}$$

Proof. The number of branches in the brush is $f = 1 - t + \sum_{i=1}^{t} v_i$. We obtain an upper bound on $b_n(t, v_1, v_2, \ldots, v_t, L)$ by a suitable concatenation of f n-step walks in an L-slab to give

$$b_n(t, v_1, v_2, \ldots, v_t, L) \leq c_n(L)^f \tag{2.7}$$

where $c_n(L)$ is the number of n-step self-avoiding walks in an L-slab. Two walks are counted as distinct if they cannot be superimposed by translation. Note that $\lim_{n\to\infty} n^{-1} \log c_n(L) = \kappa(L)$. To obtain a lower bound we concatenate a walk unfolded in the x direction in an L-slab and a sequence of t uniform stars, each unfolded in an L-slab, having $v_1 - 1, v_2 - 1, \ldots, v_t - 1$ branches. With the first unfolded star fixed with its branch point at $(0, 0, 1)$ we add the edge $(0, 0, 1)$–$(-1, 0, 1)$ and an $(n-1)$-step unfolded walk, starting at $(-1, 0, 1)$ and reflected in the plane $x = -1$. The concatenation of the kth and $(k+1)$th stars is accomplished by identifying the rightmost vertex of the kth unfolded star and the branch point of the $(k+1)$th unfolded star. The resulting graphs are uniform brushes in an L-slab and

$$b_n(t, v_1, v_2, \ldots, v_t, L) \geq c_n^+(L, 1) \prod_{i=1}^{t} s_n^+(v_i - 1, L, z_i) \tag{2.8}$$

where $z_1 = 1$ and $z_2, z_3, \ldots, z_t$ are fixed by the concatenation and where $c_n^+(L, 1)$ is the number of walks unfolded in the x direction in an L-slab and starting at the point $(0, 0, 1)$. This bound, together with equation (2.7), proves the theorem.

A similar construction for the complete lattice establishes that uniform brushes have the same connective constant as walks in $\mathbb{Z}^3$, and a similar argument for the complete lattice works in $\mathbb{Z}^d$.

3. Brushes in a slit

In this section we investigate the connective constant of uniform brushes in an L-slit in $\mathbb{Z}^2$. The main result is that the connective constant of a brush in an L-slit is strictly less than that of a self-avoiding walk in an L-slit.

We now write $b_n(t, v_1, v_2, \ldots, v_t, L)$ for the number of uniform brushes in an L-slit, having t branch points, of vertex degree $v_1, v_2, \ldots, v_t$, and n edges in each branch. Two brushes are counted as distinct if they cannot be superimposed by translation.

We have been unable to prove the existence of the limit $\lim_{n\to\infty} [n(1 - t + \sum_{i=1}^{t} v_i)]^{-1} \log b_n(t, v_1, v_2, \ldots, v_t, L)$ and our results take the form of bounds on the corresponding limiting infimum and limiting supremum. We now write $c_n(L, y_0)$ for the number of n-step self-avoiding walks confined to an L-slit, starting at $(0, y_0)$, for any y_0 such that $0 \leq y_0 \leq L$. $c_n^+(L, y_0)$ is the number of n-step self-avoiding walks confined to an L-slit and unfolded in the x direction, starting at $(0, y_0)$, for any y_0 such that $0 \leq y_0 \leq L$. $c_n(L)$ is the number of n-step self-avoiding walks confined to an L-slit such that two walks are counted as distinct if they cannot be superimposed by translation. The three limits, $\lim_{n\to\infty} n^{-1} \log c_n(L, y_0)$, $\lim_{n\to\infty} n^{-1} \log c_n^+(L, y_0)$ and $\lim_{n\to\infty} n^{-1} \log c_n(L)$, are known to exist and to be equal. We define $\kappa(L)$ to be the value of these limits.

Theorem 3.1. For any $t, v_1, \ldots, v_t$, such that $v_i \leq 4$ for all i,

$$\limsup_{n\to\infty}(nf)^{-1}\log b_n(t, v_1, v_2, \ldots, v_t, L) \leq \frac{t+1}{f}\kappa(L) + \frac{f-t-1}{f}\kappa(L-1) < \kappa(L) \tag{3.1}$$

where $f = 1 - t + \sum_{i=1}^{t} v_i$.

Proof. We define the top (bottom) vertex of a brush as the vertex having largest (smallest) y coordinate in the subset of vertices of the brush having largest (smallest) x coordinate. Let the top vertex have coordinates (x_t, y_t) and the bottom vertex have coordinates (x_b, y_b). The brush lies in, or in the boundary of, the rectangle R with vertices $(x_b, -\frac{1}{2})$, $(x_b, L+\frac{1}{2})$, $(x_t, L+\frac{1}{2})$, $(x_t, -\frac{1}{2})$. There is a self-avoiding walk w, which is a subgraph of the union of at most $t+1$ branches of the brush, from (x_b, y_b) to (x_t, y_t), and there is a subwalk (w') of w which has two vertices of degree one in ∂R, the boundary of R, but no edge in ∂R. $\partial w'$ is properly embedded in ∂R and $R - w'$ is not connected since w' separates R into two components. Hence there are at least $f-t-1$ *interior* branches of the brush, each of which have all their edges in a component of $R - w'$. All points on the line $y = L$ between (x_b, L) and (x_t, L) lie above or on w', so that one of the components of $R - w'$ lies entirely below the line $y = L$. Similarly, the other component of $R - w'$ lies entirely above the line $y = 0$. Therefore all, except possibly the first step, of each interior branch must lie entirely in a slit of width $L - 1$. Hence

$$b_n(t, v_1, v_2, \ldots, v_t, L) \leq c_n(L)^{t+1} \prod_{i=1}^{f-t-1} c_{m_i}(L-1) \tag{3.2}$$

where $m_i = n$ or $n - 1$. Taking logarithms, dividing by fn and letting $n \to \infty$ proves the theorem.

There are several approaches to the derivation of lower bounds to the number of brushes in an L-slit and we illustrate some of these in the following lemmas.

We first establish an inequality between the number of uniform combs and the number of polygons in an L-slit, $p_n(L)$. The connective constant

$$\kappa_0(L) \equiv \lim_{n\to\infty} n^{-1}\log p_n(L) \tag{3.3}$$

is known to exist (Soteros and Whittington 1988) and numerical values of $\kappa_0(L)$ are available for small L (Klein 1980).

Lemma 3.1. Let $r_n(t, L) = b_n(t, 3, 3, \ldots, 3, L)$ be the number of uniform combs in an L-slit with t teeth and n edges in each of the $f = 2t + 1$ branches. Then

$$\liminf_{n\to\infty}(nf)^{-1}\log r_n(t, L) \geq [\kappa(L) + 2t\kappa_0(L)]/f. \tag{3.4}$$

Proof. We define an L-pad to be a rectangular polygon of $2L + 2$ edges, one edge in each of the lines $y = 0$ and $y = L$. We concatenate an L-pad and a polygon in an L-slit with $2n - 2L - 2$ edges by translating the polygon so that its bottom vertex is one lattice space to the right of the L-pad. Delete the edge (x_b, y_b)–$(x_b, y_b + 1)$ in the polygon and the edge $(x_b - 1, y_b)$–$(x_b - 1, y_b + 1)$ in the L-pad and add the edges

$(x_b - 1, y_b)$–(x_b, y_b) and $(x_b - 1, y_b + 1)$–$(x_b, y_b + 1)$. We call this a *padded polygon* with $2n$ edges.

We concatenate a walk unfolded in the x direction in an L-slit and a sequence of t padded polygons, as follows. We concatenate an $(n - 1)$-step self-avoiding walk, starting at the origin, confined to an L-slit and unfolded in the x direction, and a padded polygon with $2n$ edges, by adding a step in the x direction joining the end point of the walk to the pad. This creates the first branch point of degree 3. The concatenation of the kth and $(k+1)$th padded polygons proceeds as follows. Identify the vertex, (x_n, y_n), which divides the kth padded polygon into two n-step walks, w_1 and w_2, starting at the kth branch point. If w_1 (w_2) contains the top vertex of the padded polygon, remove the last step of w_1 (w_2) and unfold w_1 (w_2) in the positive x direction; in case of ambiguity (i.e. $(x_t, y_t) = (x_n, y_n)$) remove the edge (x_n, y_n)–$(x_n, y_n - 1)$ and unfold the walk containing the vertex $(x_n, y_n - 1)$ in the positive x direction. Add a step in the positive x direction to the unfolded walk and join its endpoint to the pad of the $(k+1)$th padded polygon to create the $(k+1)$th branch point of degree 3. See figure 2 for an example. The resulting graph is a uniform comb in an L-slit and hence

$$r_n(t, L) \geq c^+_{n-1}(L, 0)(\exp(O(\sqrt{n}))p_{2n-2L-2}(L))^t \tag{3.5}$$

which with equation (3.3) gives equation (3.4).

Figure 2. (*a*) shows a 19-step unfolded walk and a sequence of three 40-step padded polygons. These can be concatenated as in (*b*) to give a uniform comb with 20 steps in each branch.

For the subset of the brushes in which each brush contains at least one vertex of degree 4 (i.e. the brush is not a comb) we can concatenate in a similar fashion a polygon and a sequence of uniform stars and padded polygons.

Consider a uniform star with f branches each containing n edges, having its branch point at $(0, y_0)$, $0 \leq y_0 \leq L$ and write (x_{ij}, y_{ij}) for the coordinates of the ith vertex in the jth branch. Replace z_{ij} by y_{ij} in the definition of a star unfolded in an L-slab to obtain the definition of a 3-star *unfolded in an L-slit* and let $s^+_n(3, L, y_0)$ be the number of such unfolded stars.

Lemma 3.2. For any t, $v_1, \ldots, v_t$, such that $v_i \leq 4$ for all i and assuming there exists m such that $v_m = 4$:

$$b_n(t, v_1, \ldots, v_t, L) \geq \exp(O(\sqrt{n}))p_{2n-2}(L)p_{2n-2L-4}(L)p_{2n-2L-2}(L)^{n_3} \prod_{i=1}^{n_4-1} s^+_n(3, L, y_o(i)) \tag{3.6}$$

where n_k is the number of vertices of degree k in the brush.

Proof. The proof follows by a straightforward concatenation argument which we sketch without giving details. The first vertex of degree 4 in the ordered set $\{v_i\}$ is obtained by concatenating a $(2n-2)$-step polygon confined to an L-slit and a $(2n-2)$-step padded polygon. This is cut and unfolded (to the right) in a similar way to that described in lemma 3.1 and concatenated with a $2n$-step padded polygon or an unfolded 3-star with n steps in each branch, according to whether the next vertex is of degree 3 or 4. This is continued (to the right) to construct all branch points later in the list $\{v_i\}$ and then (to the left) to construct branch points earlier in the list. See figure 3 for an example.

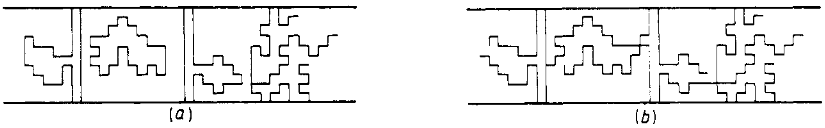

Figure 3. (*a*) A 40-step padded polygon, a 38-step polygon, a 38-step padded polygon and an unfolded 3-star with 20 steps in each arm. These can be concatenated as in (*b*), starting with the two 38-step polygons in the middle, to give a brush with $v_1 = 3$, $v_2 = v_3 = 4$.

Note that equations (3.5) and (3.6) with $t = 1$ reduce to the bounds obtained for 3-stars and 4-stars, respectively, in Soteros and Whittington (1988).

Lemma 3.3. For any $t, v_1, \ldots, v_t$, such that $v_i \leq 4$ for all i,

$$\lim_{L\to\infty} \liminf_{n\to\infty} (nf)^{-1} \log b_n(t, v_1, \ldots, v_t, L) = \kappa. \tag{3.7}$$

Proof. The strategy is to obtain a lower bound for $s_n^+(3, L, y_0)$ in terms of walks and thus obtain a lower bound for $\liminf_{n\to\infty} (nf)^{-1} \log b_n(t, v_1, \ldots, v_t, L)$ using equation (3.6).

We divide the slit of width L into three sub-slits of width $\lfloor (L-1)/3 \rfloor$, $\lfloor (L-1)/3 \rfloor$, $L-2-2\lfloor (L-1)/3 \rfloor$, such that an adjacent pair of slits is separated by one lattice space. ($\lfloor x \rfloor$ denotes the greatest integer smaller than or equal to x.) By considering unfolded walks in each of these three slits, and concatenating them to form an unfolded 3-star one obtains

$$\liminf_{n\to\infty} (3n)^{-1} \log s_n^+(3, L, y_0) \geq \tfrac{2}{3}\kappa(\lfloor (L-1)/3 \rfloor) + \tfrac{1}{3}\kappa(L-2-2\lfloor (L-1)/3 \rfloor). \tag{3.8}$$

Then equation (3.8) together with equations (3.3) and (3.6) gives

$$\begin{aligned} \liminf_{n\to\infty} & (nf)^{-1} \log b_n(t, v_1, \ldots, v_t, L) \\ & \geq ((2n_3+4)/f)\kappa_0(L) + ((3n_4-3)/f)(\tfrac{2}{3}\kappa(\lfloor (L-1)/3 \rfloor) \\ & \quad + \tfrac{1}{3}\kappa(L-2-2\lfloor (L-1)/3 \rfloor)) \end{aligned} \tag{3.9}$$

when $n_4 \geq 1$. By concatenating pairs of walks in slits of width $\lfloor (L-1)/2 \rfloor$ to form polygons in a slit of width L it is straightforward to obtain a lower bound on $\kappa_0(L)$ in terms of $\kappa(\lfloor (L-1)/2 \rfloor)$ from which it follows that

$$\lim_{L\to\infty} \kappa_0(L) = \kappa \tag{3.10}$$

and so

$$\lim_{L\to\infty} \liminf_{n\to\infty} (nf)^{-1} \log b_n(t, v_1, \ldots, v_t, L) \geq ((2n_3+3n_4+1)/f)\kappa = \kappa. \tag{3.11}$$

We obtain a similar result from equation (3.4) when $n_4 = 0$. Then using equation (3.1) we have

$$\lim_{L\to\infty} \liminf_{n\to\infty} (nf)^{-1} \log b_n(t, v_1, \ldots, v_t, L) = \kappa. \tag{3.12}$$

A similar result applies to the $L \to \infty$ limit of the $n \to \infty$ limiting supremum.

4. Brushes in a prism

In this section we investigate the connective constant of walks, polygons and brushes in an infinite right-rectangular prism. An (L_1, L_2)-prism is that part of $\mathbb{Z}^3$ lying in and between the four planes $y = 0$, $y = L_1$, $z = 0$, $z = L_2$. The main results are that the connective constant of a brush and a polygon are strictly less than that of a self-avoiding walk in an (L_1, L_2)-prism.

We now write $c_n(L_1, L_2)$, $p_n(L_1, L_2)$, $b_n(t, v_1, \ldots, v_t, L_1, L_2)$, respectively for the number of n-step self-avoiding walks, n-step polygons and uniform brushes having t branch points, of vertex degree $v_1, \ldots, v_t$, and n edges in each branch, confined to an (L_1, L_2)-prism. Two walks, polygons or brushes are counted as distinct if they cannot be superimposed by translation. Following the proof for walks in $\mathbb{Z}^3$ (Hammersley 1957) the limit $\lim_{n\to\infty} n^{-1} \log c_n(L_1, L_2)$ can be shown to exist and is defined to be $\kappa(L_1, L_2)$. Two polygons can be concatenated via two walks (with lengths less than $2(L_1 + 1)(L_2 + 1)$) contained in a pad with dimensions $1 \times L_1 \times L_2$ and an argument similar to that in Soteros and Whittington (1988) shows that the connective constant $\kappa_0(L_1, L_2) \equiv \lim_{n\to\infty} n^{-1} \log p_n(L_1, L_2)$ exists. As in the case of a brush in an L-slit, however, we have been unable to prove the existence of the limit $\lim_{n\to\infty} (nf)^{-1} \log b_n(t, v_1, \ldots, v_t, L_1, L_2)$ for $f = 1/[n(1 - t + \sum_{i=1}^{t} v_i)]$ and our results for brushes again take the form of bounds on the corresponding limiting infimum and limiting supremum.

In order to prove our main results we need an extension of Kesten's 'pattern theorem'. For this purpose we define a $\mathscr{K}_\beta$-pattern, for any $\beta > 0$, to be any self-avoiding walk ω for which there exists a self-avoiding walk ω^*, such that ω occurs in ω^* and ω^* begins at $(0,0,0)$, ends at $(\beta, 0, 0)$ (or begins at $(\beta, 0, 0)$, ends at $(0,0,0)$) and is completely contained within the box $D_\beta = \{(x, y, z) \in \mathbb{Z}^3 : 0 \leq x \leq \beta, 0 \leq y \leq L_1, 0 \leq z \leq L_2\}$. The required extension of theorem 1 of Kesten (1963) is as follows.

Lemma 4.1. For any $\beta > 0$, let P be a $\mathscr{K}_\beta$ pattern, other than the unit step u_1, then

$$\lim_{n\to\infty} n^{-1} \log c_n(\bar{P}, L_1, L_2) < \kappa(L_1, L_2) \tag{4.1}$$

where $c_n(\bar{P}, L_1, L_2)$ is the number of n-step self-avoiding walks in an (L_1, L_2)-prism which do not contain the pattern P. These walks are counted as distinct if they cannot be superimposed by translation.

Proof. We prove first that the limit $\lim_{n\to\infty} n^{-1} \log c_n(\bar{P}, L_1, L_2)$ exists. Disconnecting a walk cannot form a pattern P. Hence, $c_{m+k}(\bar{P}, L_1, L_2) \leq c_m(\bar{P}, L_1, L_2) c_k(\bar{P}, L_1, L_2)$ and since $1 \leq c_n(\bar{P}, L_1, L_2)^{1/n}$, the limit

$$\kappa(\bar{P}, L_1, L_2) \equiv \lim_{n\to\infty} n^{-1} \log c_n(\bar{P}, L_1, L_2) \tag{4.2}$$

exists and is non-negative.

We say the event E occurs at the rth step of a walk ω if for some α with $-\beta \leq \alpha \leq 0$ all points of the box $D_\beta(r, \alpha) = \{(x, y, z) \in \mathbb{Z}^3 : \alpha \leq x - x_r(\omega) \leq \beta + \alpha, 0 \leq y \leq L_1, 0 \leq z \leq L_2\}$ are occupied by ω, where $x_j(\omega)$ denotes the x coordinate of the jth step of ω. Following the proof of lemma 5 of Kesten (1963) it can be shown that $\lim_{n\to\infty} n^{-1} \log c_n(\bar{E}, L_1, L_2) < \kappa(L_1, L_2)$, where $c_n(\bar{E}, L_1, L_2)$ is the number of n-step walks in which E never occurs. Hence, the event E occurs in almost all (i.e. all except exponentially few) walks in an (L_1, L_2)-prism. Following a proof similar to that of lemma 4 of Kesten (1963), it can be shown that E occurs in almost all walks of length n, n large, at least ϵn times for some ϵ. Whenever the event E occurs in a walk it can be replaced by the pattern P. This is the essence of the proof of theorem 1 of Kesten (1963) and a similar proof applies in this case. In fact it is possible to show that there exists $\epsilon > 0$ such that $\limsup_{n\to\infty} n^{-1} \log c_n(\epsilon n, P, L_1, L_2) < \kappa(L_1, L_2)$ where $c_n(\epsilon n, P, L_1, L_2)$ is the number of walks in which P occurs at most ϵn times.

Theorem 4.1.

$$\kappa_0(L_1, L_2) < \kappa(L_1, L_2). \tag{4.3}$$

Proof. Let ω_1 be the walk which completely fills the set $\{(0, y, z) \in \mathbb{Z}^3 : 0 \le y \le L_1, 0 \le z \le L_2\}$ so that the walk starts at $(0, 0, 0)$ and its position after the jth step is $(x_j, y_j, z_j) = (0, \lfloor j/(L_2+1)\rfloor, L_2/2 + (-1)^{\lfloor j/(L_2+1)\rfloor}(-L_2/2 + j - (L_2+1)\lfloor j/(L_2+1)\rfloor))$ for $0 \le j \le (L_1+1)L_2 + L_1$. The walk after $(L_1+1)L_2 + L_1$ steps ends at the vertex $(0, L_1, (L_2/2)(1 + (-1)^{L_1}))$. If L_1 is odd, add to ω_1 the step to $(1, L_1, 0)$ and consecutively the steps to $(1, L_1 - j, 0)$ for $j = 1, \ldots, L_1$. If L_1 is even, add to ω_1 the step to $(1, L_1, L_2)$, consecutively the steps to $(1, L_1, L_2 - k)$, $k = 1, \ldots, L_2$, and consecutively the steps to $(1, L_1 - j, 0)$, $j = 1, \ldots, L_1$. In both these cases add the step from $(-1, 0, 0)$ to $(0, 0, 0)$ to create a $\mathcal{K}_2$ pattern starting at $(-1, 0, 0)$ and ending at $(1, 0, 0)$ which we call the *filling pattern* for an (L_1, L_2)-prism. See the sketches in figure 4.

Figure 4. (*a*) and (*b*) show the filling pattern in the case that L is odd and even, respectively.

A polygon in an (L_1, L_2)-prism can be divided into two subwalks which start at the bottom vertex (x_b, y_b, z_b) and end at the top vertex (x_t, y_t, z_t). (Top and bottom vertices in a prism can be defined in a way similar to the definition given for an L-slit.) Since these two walks are disjoint apart from their endpoints neither of them can contain the filling pattern. Therefore

$$p_n(L_1, L_2) \le \sum_{m=1}^{n-1} c_{n-m}(\bar{P}, L_1, L_2) c_m(\bar{P}, L_1, L_2) \tag{4.4}$$

where P is the filling pattern. Then equation (4.4) and lemma 4.1 gives the theorem.

Theorem 4.2. For any t, $v_1, \ldots, v_t$, such that $v_i \le 6$ for all i,

$$\limsup_{n\to\infty} (nf)^{-1} \log b_n(t, v_1, \ldots, v_t, L_1, L_2) < \kappa(L_1, L_2). \tag{4.5}$$

Proof. For any brush, there is a self-avoiding walk ω, which is a subgraph of at most $t+1$ branches of the brush, starting from (x_b, y_b, z_b) and ending at (x_t, y_t, z_t). There are at least $f - t - 1$ *interior* branches of the brush which have no edges in ω. If one of these interior branches contains the filling pattern then this branch may intersect ω at the branch point vertex and will intersect ω in at least one other vertex which will be a part of the filling pattern. Thus no interior branch can contain the filling pattern and therefore

$$b_n(t, v_1, \ldots, v_t, L_1, L_2) \leq c_n(L_1, L_2)^{t+1} c_n(\bar{P}, L_1, L_2)^{f-t-1} \tag{4.6}$$

where P is the filling pattern. Lemma 4.1 and equation (4.6) together give the theorem. Putting t=1 in equation (4.5) gives a bound for stars, and putting $v_1 = v_2 = \ldots = v_t = 3$ in equation (4.5) gives a bound for combs.

5. Discussion

We have considered the effect of various geometrical constraints (confinement to a slab, slit or prism) on the connective constant of uniform brushes (and stars and combs) weakly embeddable in a lattice. In a slab geometry the connective constant of a brush is identical to that of a self-avoiding walk in a slab. In a slit or prism the connective constant of a brush is less than that of a walk, which means that a brush loses more configurational entropy than a walk when confined in this way. We have also confirmed and extended an observation by Klein (1980) that a polygon in a prism has a lower connective constant than a walk in a prism.

Perhaps the most interesting aspect of these results is the difference in behaviour for a pseudo-two-dimensional constraint (a slab) and a pseudo-one-dimensional constraint (a slit or prism). Our proof for the slit case makes use of a version of the Jordan curve theorem which emphasises the essential topological distinction between $\mathbb{R}^2$ and $\mathbb{R}^3$. This proof does not work for a prism since a closed curve does not divide a prism regarded as a subset of $\mathbb{R}^3$. However, since we are concerned with subsets of $\mathbb{Z}^3$, it is possible to divide a prism in $\mathbb{Z}^3$ by an appropriately defined filling pattern. This is the essence of the proof presented in section 4. A similar proof can be constructed for $\mathbb{Z}^2$.

Finally we point out several open questions. It would be useful and interesting to supply a proof of the existence (or otherwise!) of the connective constant for uniform brushes in a slit or prism. There are also no numerical estimates of this quantity and we do not know how good our lower bounds are. In particular, the bound presented in lemma 3.1 for uniform combs in a slit of width L, combined with Klein's estimates for $\kappa(L)$ and $\kappa_0(L)$ gives numerical bounds on the connective constant of combs and these are given in table 1.

Table 1. Lower bounds on the connective constant of combs in a slit.

L	$t = 2$	$t = 3$	$t = 4$
2	0.4071	0.3898	0.3802
3	0.5631	0.5507	0.5439
4	0.6554	0.6459	0.6407

It would be interesting to compare this with numerical results from e.g. Monte Carlo or transfer matrix calculations. Certainly this lower bound is exact when $L = 1$.

The same bound works for a comb in a prism and, using Klein's results for $\kappa(1, 1)$ and $\kappa_0(1, 1)$, we obtain 0.6579 as a lower bound on the connective constant for uniform combs with two teeth in the (1, 1)-prism. Similar numerical bounds can easily be written down for larger values of t.

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